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principle in its simplest form. Sometimes the strokes are arranged in rows of ten or every tenth stroke is made longer than the others. Instances of this kind are found for the expression of numbers as high as thirty. They are found among the more highly civilized Indians of the middle west, especially among the Dakotas. On the other hand the Comanches are said to have been "ignorant of the elements of figures, even of a perpendicular stroke for one."[1] Pure numeral notation is not always found. Frequently the number of objects is expressed by the repetition of the symbol for the object the desired number of times, especially in the case of men or tepees. Another method is by dots. A man's head with eight dots above it in one case means nine men, the head itself counting for one. Another picture gives a head over which are thirty black dots in three lines of ten each. This is said to mean thirty men, not thirty-one. Thus the usage is not fixed. The use of notches cut on sticks was frequent, not only in the middle west but in California and on Puget Sound in western Washington. At the San Gabriel mission in California every tenth notch was cut entirely across the stick instead of only in the corner.

(*To be continued.*)

SYNTHETIC PROJECTIVE GEOMETRY AS AN UNDERGRADUATE STUDY.

By W. H. BUSSEY, University of Minnesota.

Modern Geometry.—Synthetic projective geometry is called modern synthetic geometry because the theories and methods that make of its propositions a homogeneous and harmonious whole are due for the most part to men of a time near our own, namely Poncelet, Möbius, Steiner, Chasles, Von Staudt, Reye, Cremona, and others. Some of its theorems can be traced back to Euclid, Apollonius, and Pappus, but it is in the work of Pascal and Desargues that the germs of the modern theories are to be found. When he was only sixteen years old Pascal discovered his famous theorem about a hexagon inscribed in a conic and made it the basis of a treatise on conics which is unfortunately lost. It was never published but we know something of its contents and extent from a report made by Leibnitz who saw it in Paris. Two of the most important theorems of projective geometry go by the name of Desargues. One is that if any two triangles in a plane or in space are such that corresponding vertices are collinear in pairs with a fixed point, then the corresponding sides are concurrent in pairs with a fixed straight line. The other is that any transversal meets a conic and the pairs of opposite sides of any inscribed quadrangle in points of an involution. But unfortunately for the development of the subject the work of Pascal and Desargues was overshadowed by the geometry of their contemporary Descartes. Analytic geometry and the calculus, invented about thirty years later, became the chief instruments of progress in geometry as in all mathematics, and the work of Pascal and Desargues lay undeveloped for nearly 200 years. The revival of synthetic geometry by methods similar to theirs was due chiefly to Poncelet, and for that reason the origin of projective geometry is usually ascribed to him.

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[1] Eakins, D. W.: in Schoolcraft's *Indian Tribes*, Philadelphia, 1851–60, Vol. 1, p. 416.

Synthetic versus Analytic Geometry.—Synthetic projective geometry is still overshadowed by analytic geometry and the calculus and it does not have the place in the undergraduate curriculum that it ought to have. Indeed it may be said that it has no recognized place, for in most American colleges and universities it is not offered at all, and there is nothing like uniformity in the conditions under which it is offered in those that do give a course in the subject. In some colleges it is primarily for graduates and in others it is primarily for undergraduates. More often it is for both undergraduates and graduates. The subject is offered three hours a week for a semester, two hours a week for a year, and three hours a week for a year. A number of colleges are offering an introductory course as an elective to juniors and seniors who have had a first course in the calculus. In a few cases the only prerequisite is a first course in analytic geometry and the course is open to sophomores. When given under these conditions the course must of necessity be very different from that given in some institutions to graduate students who have a reading knowledge of French and German. At Bryn Mawr a lecture course in the subject is offered as a post-major course to juniors and seniors who have taken college mathematics five times a week for two years, and a reading knowledge of French and German is a prerequisite.

The subject if properly taught is well within the grasp of the undergraduate who has had a first course in the calculus. An introductory course of at least three hours a week for a semester should be offered to such students in every college, and those preparing to teach mathematics in the high school should be encouraged to take it. The subject is so recommended at Vassar, the prerequisite being analytic geometry. It is a significant fact that most of the prominent colleges for women offer courses in synthetic projective geometry to junior and senior undergraduates. Bryn Mawr, Mount Holyoke, Smith, Vassar, Wellesley, Wells, and Goucher College of Baltimore do so. The course at Bryn Mawr requires considerable mathematical maturity and a reading knowledge of French and German, but at the other colleges the course is less advanced. At two of them the prerequisite is analytic geometry.

Curriculum Now Planned for the Prospective Graduate Student.—The reason why projective geometry has no well-defined place in the undergraduate curriculum is not hard to find. The curriculum is planned primarily for those who are to go on with graduate work in mathematics or physics. It takes two years, sometimes two and a half, for the student to get through college algebra, trigonometry, analytic geometry, and a year of the calculus. Then if he is to be well prepared for graduate work he ought to take advanced calculus, solid analytic geometry, and a course in determinants and the theory of equations; a second course in plane analytic geometry is needed for a working mastery of the subject, and he ought to have at least two years of undergraduate physics and one of astronomy, and perhaps a year of theoretical mechanics. He is not likely to have time for any more electives. In fact most graduate students have not had as good a preparation as this. One or more of these subjects is likely to be left for the first graduate year. But every one of the subjects is of more importance to the prospective graduate student than synthetic projective geometry, because they are all on the main line of progress in mathematics and are all prerequisites for advanced work in applied mathematics. Projective geometry is more an end in itself. It is not needed so much as a prerequisite for something else. It is not a prerequisite for graduate work in mathematical physics or astronomy. So it is left out by most prospective graduate students even if they have an opportunity to get it as undergraduates, as most of them do not. They get it as graduate students or they do not get it at all. Then they are more mature and the course can be made, and usually is made, to cover much more ground and to be more intensive than an undergraduate introductory course in the subject. A reading knowledge of French and German can be assumed, and there need be no hesitancy in making the course one in pure geometry. It can be as abstract as the teacher pleases and it ought to involve to a greater or less extent a study of the foundations of geometry. Properly qualified seniors could take such a course, but it is not suitable for those who have just completed a first course in the calculus.

Part of a Liberal Education.—The students who take elective courses in mathematics after having had a first course in the calculus are of three kinds; those who are preparing to do graduate work in mathematics, physics, or astronomy; those who are going to college for a liberal education and who are taking mathematics because they like it and not because they expect to use it; and those who are taking mathematics in college as a preparation for teaching mathematics in the high school. For reasons already given a student of the first kind can afford to wait and take projective geometry when he is a graduate. If he is interested primarily in physics or astronomy, he may not want it at all. A student of the second kind is prepared by his one year's work in the calculus for the undergraduate courses in physics and astronomy that he may wish to take. Any other courses in mathematics that he elects he will choose on account of their interest to him. He may be one of those who always liked geometry better than algebra, and there are many such. If he belongs to this class, a course in modern synthetic geometry would probably be more interesting to him than any other course that could be offered. At any rate he ought to have a chance to take it if he wants to.

Preparation for Teaching.—These remarks about the student who is taking mathematics as a part of a liberal education apply equally well to the student of the third kind who is taking mathematics as a preparation for teaching. In many institutions this class is the largest of the three. In coeducational colleges it consists mostly of young women. College algebra, trigonometry, analytic geometry, and the calculus are a better preparation for teaching algebra than they are for teaching geometry. They involve almost constant algebraic manipulation even when their subject matter is not algebra. It is often said that a student is not really a master of high school algebra until he has been through a course in the calculus. Many prospective high school teachers take, in addition to these subjects, a course in determinants and the theory of equations and thereby make their preparation for teaching algebra so much the better. But what preparation for teaching geometry does the prospective high school teacher get that is at all comparable with the preparation for teaching algebra afforded by these courses? Of course all courses in mathematics are indirectly good preparation for teaching geometry. Experience in one branch of mathematics counts in the study of any other branch to some extent, but that does not alter the fact that, as it now stands, the undergraduate curriculum in mathematics affords a better preparation for teaching algebra than geometry. Algebra and analysis have the right of way. The geometry of the high school is synthetic. It consists largely of memory work and deductive reasoning. It holds its place in the high school not only because it is a useful subject in the work of the world but also because it is a good course in logical reasoning. What better preparation for teaching it can the student get than a course in synthetic projective geometry? It is more closely related to the geometry of Euclid than any other branch of mathematics. It gets the student back to synthetic methods. It involves more of reasoning and less of the technique of manipulation than either analytic geometry or the calculus, and it can be taught so as to be as easily within the mental grasp of the college sophomore as either of these subjects. Construction problems with the ruler and compass, with the ruler alone, with the graded ruler, and with the parallel ruler should be an important part of the introductory undergraduate course in synthetic projective geometry. There is a wealth of such problems in Cremona's *Elements of Projective Geometry*, and the teacher can find more, if they are wanted, in Adler's *Geometrische Konstruktionen.* To draw a line through the inaccessible point of intersection of two given lines by means of the ruler alone, to draw a line through a given point and parallel to a given line by means of the graded ruler or the parallel ruler, and the many other constructions possible by the principles of projective geometry are fascinating to lovers of geometry, and the constructions that can be made by means of the theorems of Pascal, Brianchon, and Desargues are a revelation to the student. What better preparation is there than this for teaching that part of the high school course which deals with construction problems?

Generality and Power of Synthetic Geometry.—Many a student leaves college to become a teacher of high school geometry with the notion that no progress in geometry is possible except by means of coördinates and algebra, and that there is no higher geometry more closely related to the geometry of Euclid. This ought not to be so. The course in modern synthetic geometry would not be so interesting to the student nor so valuable in broadening his mental horizon if it were merely a sequel to Euclid. It is more than that, inasmuch as it is characterized by the great generality and power of its methods and theorems. It is said that Pascal, in his lost treatise on conics, had 400 corollaries to his famous theorem of the mystic hexagram. The method of projection whereby theorems about circles are translated into theorems about conics; the principle of duality whereby every theorem about the incidence of points, lines, and planes yields another about planes, lines, and points; and the principle of continuity, which Chasles prefers to call the principle of contingent relations,—these are very general in their applications and are of great power in research. There is nothing to match them in the geometry of the Greeks. Their geometry consists for the most part of isolated theorems. They were more anxious to convince than to explain, and the student gets no information about the way in which the various theorems were discovered. In modern synthetic geometry, on the other hand, the student is to a considerable extent led to understand how many of the theorems came to be discovered. A good example of the great power of the principle of continuity is afforded by the long list of deductions that can be made from the theorems of Pascal and Brianchon. The student can discover some of them for himself as soon as he is let into the secret of the method. The principle of continuity indicates what the theorems are. Afterwards they can be proved rigorously. This modern synthetic geometry is very different from the synthetic geometry of the Greeks both in subject matter and in method, but it has enough in common with it to make it a good preparation for the teaching of high school geometry.

One Semester for the First Course.—The college student who is preparing to be a high school teacher must be qualified to teach more than one subject, at least in the beginning of his teaching career. Teachers in small high schools sometimes have to teach four or five. Certain courses in psychology and education are required for state teachers' certificates, and it is expected that every high school teacher will have a good liberal education. The student who is going to be a teacher with mathematics as a specialty ought to take courses in college physics and astronomy. For these reasons the prospective teacher often finds that he cannot take more than one or two half year courses in mathematics after he has had a year of the calculus. In many colleges the second half year of the calculus is normally taken in the first half of the student's junior year. Synthetic projective geometry ought to be available for these people, many of whom will be good high school teachers of geometry although they would not all, or even nearly all, be successful in graduate work in mathematics. But the course should not be so arranged that the student must take a whole year of the subject. It should be a half year course, or if a year's work is offered it should be so arranged that the work of the first semester constitutes a good course by itself, just as the first course in analytic geometry is one that is worth while even if the student never takes any more of it. A good introduction to projective geometry can be given in a half year, as in the case of analytic geometry. Then the second half year's work can be thought of as a second course for those who want it and have time to take it. It will be on a par with the second course in analytic geometry as far as its relation to the first course is concerned. In some colleges a half year of synthetic projective geometry is followed by a half year of advanced analytic geometry, or vice-versa, neither being a prerequisite for the other. Sometimes the two are offered in alternate years so that the student who wishes can get either or both. This plan is necessary in some colleges because there is not enough demand for both courses every year. Another plan is to give a year's course in modern geometry, synthetic and analytic methods both being used all the way through the course. This makes a good year's course but is not so well adapted to the needs of the student who can take only one semester's work. This plan is followed in a number of colleges, but many times the course announced as modern geometry, whether it is a year or a half year course, is too exclusively taken up with the analytic point of view at the expense of synthetic projective geometry. This is due to some extent to the fact that many college teachers of mathematics have never had an adequate training in modern synthetic geometry, and consequently their interest is on the analytic side. This is another consequence of the fact that projective geometry is not on the main line of progress in mathematics. Thus the present status of modern synthetic geometry tends to perpetuate itself.

A Brief Outline of a Suitable Course.—The introductory course of a half year three times a week, or its equivalent, should not be a course in pure geometry that avoids the measurement of geometric magnitudes and all metric properties, like the geometry of Von Staudt. Nor should it be a study of the foundations of geometry, or involve such a study, to any considerable extent. The plane geometry of the high school should be assumed in its entirety just as it is for the first course in analytic geometry. The course will from its very nature deal largely with projective rather than metric properties of geometrical figures, but to avoid all metric notions is not wise in a course open to college juniors. Anharmonic ratios should be used freely, and the measurement of geometric magnitudes is involved in their definition.

The following is a brief outline of a course that can be completed in a half year, if not too much time is given to any one of the topics. The author of this paper has tried it with satisfactory results:

> Central projection and figures in perspective, points at infinity, the principle of duality, projective geometric forms, harmonic forms, anharmonic ratios, the construction of geometric forms, projective forms in relation to the circle, projective forms in relation to the conic sections, the theorems of Pascal and Brianchon and their many corollaries, the principle of continuity, projective ranges on a conic and projective series of tangents to a conic, superposed projective ranges and involutions, the theorem of Desargues about a conic and an inscribed quadrangle, problems of the second degree, pole and polar, centers and diameters. Construction problems to be given all the way through the course, as many as there is time for.

The Question of a Suitable Text.—There is no satisfactory textbook for this course. Cremona's *Elements of Projective Geometry* can be used to advantage if it is supplemented to a considerable extent by lectures, and if good judgment is used in omitting some paragraphs and in substituting for others. The writer has used the book in this way but has found it necessary at times to substitute other proofs for the ones given by Cremona because his proofs involved paragraphs that had been omitted. Or the course can be made a lecture course with considerable work in assigned problems and outside reading. Reye's *Geometry of Position* (volume 1, English translation by T. F. Holgate) is out of print, but it is likely to be in the library so that it can be used for reference. The *Projective Geometry* by Veblen and Young is too advanced and abstract for students who have had only two years of college mathematics, and besides it is not suitable for a short course. But it can be used to advantage for reference at times. Emch's *Projective Geometry* and Miss Scott's *Modern Geometry* get at the subject mostly from the side of analytic geometry, but they can be used for reference at times. There are some English books that the teacher may want to have in the library for occasional reference, although no one of them is suitable for use as a textbook for the course, namely, Russell's *Elementary Treatise on Pure Geometry,* Milne's *Elementary Treatise on Cross Ratio Geometry,* Filon's *Projective Geometry,* Pickford's *Elements of Projective Geometry,* and Durell's *Plane Geometry for Advanced Students.*

It is to be hoped that some one will publish a book on synthetic projective geometry that will be suitable for use as a textbook for an introductory course in the subject. In the meantime the course if given will have to be conducted in one of the ways suggested above.

A NOTE ON THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS.

By C. R. MAC INNES, Princeton University.

In showing how to solve linear differential equations with constant coefficients, the usual way of finding the complementary function seems to be to guess at the various parts and then to prove that the guesses are correct. A much more satisfactory way is to define the differential operators $D - \alpha$, $D - \beta$, etc., to show that they obey the laws of addition and multiplication, and then to proceed in a straightforward way to solve the problem.

Having shown that $\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y = 0$ may be written as $(D - \alpha)(D - \beta)y = 0$, one sees that a solution of $(D - \beta)y = 0$ is a solution of the original equation since $(D - \alpha)0 \equiv 0$. Since the factors are commutative any one may come last and we can get all the parts of the complementary function except those arising from repeated factors.

If $(D - \alpha)(D - \beta)y = X$, put $(D - \beta)y = v$; then $(D - \alpha)v = X$. Integrating this in the usual way one gets

$$v = (D - \beta)y = e^{\alpha x}\int e^{-\alpha x}X\,dx.$$

Hence the rule: *To remove a factor* $(D - \alpha)$ *from the left side, multiply the right member by* $e^{-\alpha x}dx$, *integrate and multiply by* $e^{\alpha x}$.